NUMERICAL SOLUTION OF EQUATIONS WITH SECOND DERIVATIVE: CASE OF NEBULAR MECHANICS

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ABSTRACT

Paper presents simple and accurate algorithm for numerical solution of equations with second derivative. Method was tested via comparison with known analytical solutions for circular and elliptic planetary orbits.

INTRODUCTION TO NEBULAR MECHANICS

In accordance with Newton's law, solar gravity may be expressed as:

$$g = k_G \times M_\odot / r^2 = K_\odot / r^2 \quad (1)$$

Here g is acceleration of gravity, k_G is gravity constant (≈ 6.67384×10^{-11} $m^3\times s^{-2}\times kg^{-1}$: Mohr et al., 2012), $M_\odot$ is mass of Sun (≈ 1.98855×10^{30} kg: calculated from $K_\odot$), $K_\odot = k_G\times M_\odot$ is solar gravity constant (≈ 1.3271283×10^{20} m^3/s^2: calculated from Eqs. 30 and 23) and r is distance to the Sun.

The planetary orbit is located in a single plane. Because of this, one may consider two coordinates of planet, x and y. If to assume that the Sun is located at origin (x = 0, y = 0), components of acceleration, g_x and g_y, and distance to the Sun, r, are defined by:

$$r = (x^2 + y^2)^{0.5} \quad (2)$$
$$d^2x/dt^2 = g_x = -g\times x/r = -K_\odot\times\{x/r^3\} \quad (3)$$
$$d^2y/dt^2 = g_y = -g\times y/r = -K_\odot\times\{y/r^3\} \quad (4)$$

Here t is time.

ANALYTICAL SOLUTION FOR CIRCULAR ORBIT

If radius of the orbit $r = r_o$ is constant (~ 149.5978707×10^9 m for the Earth: Pitjeva and Standish, 2009; Luzum et al, 2011), the mathematic model is very simple. For exactly circular orbit, velocity $v = v_o$ is also constant (~ 29785 m/s for the Earth), and may be calculated as:

$$v_o = 2\pi r_o/\tau_s = \{K_\odot/r_o\}^{0.5} \quad (5)$$

Here π = 3.141592653589793…, and τ_s is so called sidereal year, which is period of rotation around the Sun with respect to "fixed stars" (for the Earth: 31558150 s or 365 days, 6 hours, 9 minutes, and 10 seconds: Encyclopedia Britannica):

[1] with insignificant corrections from 15.06.2015 and 23.11.2015

$$\tau_s = 2\pi(r_o^3/K_\odot)^{0.5} \quad (6)$$

Similarly, acceleration at exactly circular orbit is also constant (≈ 5.9301×10^{-3} m/s^2 for the Earth's orbit):

$$g_o = K_\odot/r_o^2 = v_o^2/r_o = 4\pi^2 r_o/\tau_s^2 \quad (7)$$

Applying x, y = 0 for coordinates of Sun, assuming contra-clock rotation and, beginning from "3 o'clock", the initial state (t = 0) for planet may be defined as:

$$x_o = r_o \quad (8)$$
$$y_o = 0 \quad (9)$$

$$v_{xo} = 0 \quad (10)$$
$$v_{yo} = v_o \quad (11)$$

$$g_{xo} = -\, g_o \quad (12)$$
$$g_{yo} = 0 \quad (13)$$

For the exactly circular orbit around the immobile Sun, velocity and its components, acceleration and its components, and coordinates of planet are then given by

$$x = r_o\cos(2\pi t/\tau_s) \quad (14)$$
$$y = r_o\sin(2\pi t/\tau_s) \quad (15)$$
$$r = r_o = (x^2 + y^2)^{0.5} \quad (16)$$

$$v_x = -\, v_o\times y/r_o = -\, v_o\times\sin(2\pi t/\tau_s) \quad (17)$$
$$v_y = \quad v_o\times x/r_o = \quad v_o\times\cos(2\pi t/\tau_s) \quad (18)$$
$$v = v_o = (v_x^2 + v_y^2)^{0.5} \quad (19)$$

$$g_x = -\, g_o\times x/r_o = -\, g_o\times\cos(2\pi t/\tau_s) \quad (20)$$
$$g_y = -\, g_o\times y/r_o = -\, g_o\times\sin(2\pi t/\tau_s) \quad (21)$$
$$g = g_o = (g_x^2 + g_y^2)^{0.5} \quad (22)$$

Here t is elapsed time (t = 0 corresponds to Eqs 8-13).

ANALYTICAL SOLUTION FOR ELLIPTIC ORBIT

The elliptic orbit may be defined by two parameters: large semi-axis, a, and eccentricity, e (see Fig. 1). Large semi-axis of the Earth's orbit is equal to 1 astronomic unit (a.u.), which is (Pitjeva and Standish, 2009; Luzum et al, 2011):

$$a = 1 \text{ a.u.} = 149.5978707\times10^9 \text{ m} \quad (23)$$

Eccentricity of the elliptic orbit, e (~ 0.01671 for the Earth's orbit), is dimensionless parameter ranging from e = 0 (for exactly circular orbit) to e ≈ 1 (for ellipse reduced to straight line).

With known a and e, one may calculate other characteristics of orbit (see Fig. 1):

$$b = a(1 - e^2)^{0.5} \quad (24)$$
$$r_a = a(1 + e) \quad (25)$$
$$r_p = a(1 - e) \quad (26)$$

Here b is small semi-axis, r_p is perifocal radius (perihelion, minimum distance to the Sun), and r_a is apofocal radius (aphelion, maximum distance to the Sun).

Other useful relations:

$$e = \{1 - (b/a)^2\}^{0.5} = \{r_a - r_p\}/\{r_a + r_p\} \quad (27)$$
$$a = (r_a + r_p)/2 \quad (28)$$
$$b = (r_a r_p)^{0.5} \quad (29)$$

The period of rotation around the Sun with respect to “fixed stars”, i.e., sidereal year, τ_s (for the Earth: 31558150 s or 365 days, 6 hours, 9 minutes, and 10 seconds: Encyclopedia Britannica) is defined by third Kepler’s law:

$$\tau_s = 2\pi(a^3/K_\odot)^{0.5} \quad (30)$$

As may be seen from Eq. (30), variety of elliptic orbits, which may be inscribed into circular orbit (applying parallel shift, see Fig. 2), all have equal periods of rotation around the Sun, independent of eccentricity.

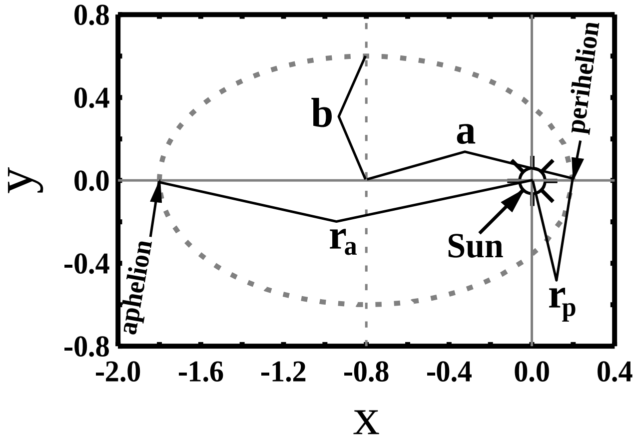

Fig. 1 Elements of elliptic orbit (at eccentricity e = 0.8): large semi-axis a = 1, small semi-axis b = 0.6, apofocal radius r_a = 1.8, perifocal radius r_p = 0.2.

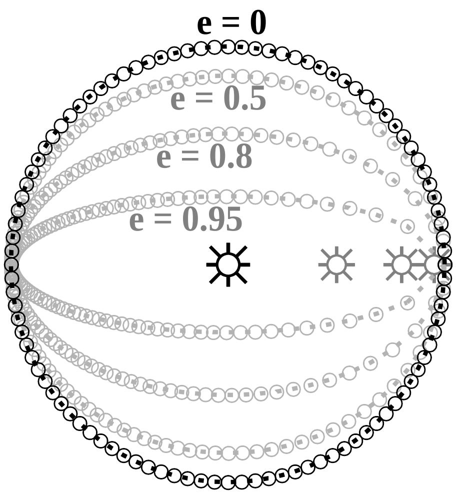

Fig. 2 Family of orbits specified by equal large semi-axis and equal periods of rotation. To match positions of perihelion and aphelion, elliptic orbits were shifted to the right; thus position of Sun shifts to the right with eccentricity.

Orbital velocity may be obtained from energy balance, which gives relation

$$v^2 = K_\odot\{2/r - (1 - e)/r_p\} = K_\odot\{2/r - 1/a\} \quad (31)$$

From Eq (31), velocity varies between minimum, v_a (in aphelion), and maximum, v_p (in perihelion):

$$v_a^2 = K_\odot(1 - e)/r_a = K_\odot\{(1 - e)/(1 + e)\}/a \quad (32)$$
$$v_p^2 = K_\odot(1 + e)/r_p = K_\odot\{(1 + e)/(1 - e)\}/a \quad (33)$$

Let us assume that the large semi axis coincides with "x" axis, the Sun is located in the right focus of ellipse, and its coordinates are x=0 and y=0 (see Fig. 1). In this case, the initial state (t = 0) may be defined by (again, assuming contra-clock rotation and, beginning from "3 o'clock"):

$$x_o = r_p = a(1 - e) \quad (34)$$
$$y_o = 0 \quad (35)$$
$$v_{xo} = 0 \quad (36)$$
$$v_{yo} = v_p = \{K_\odot(1 + e)/r_p\}^{0.5} = \{K_\odot\{(1 + e)/(1 - e)\}/a\}^{0.5} \quad (37)$$
$$g_{xo} = - K_\odot/r_p^2 = - K_\odot/\{a(1 - e)\}^2 \quad (38)$$
$$g_{yo} = 0 \quad (39)$$

In this case, coordinates of planet, velocity and its components, acceleration and its components are given by

$$x = a\times\cos(E) - ae \quad (40)$$
$$y = b\times\sin(E) \quad (41)$$
$$r = (x^2 + y^2)^{0.5} = a\{1 - e\times\cos(E)\} \quad (42)$$

$$v_x = - v\{y/b\} = - v\times\sin(E) \quad (43)$$
$$v_y = v\{x/a + e\} = v\times\cos(E) \quad (44)$$
$$v^2 = v_x^2 + v_y^2 = K_\odot\{2/r - 1/a\} = \{K_\odot/a\}\{1 + e\times\cos(Z)\}/\{1 - e\times\cos(Z)\} \quad (45)$$

$$g_x = - g\times x/r = - g(\cos(E) - e)/\{1 - e\cos(E)\} \quad (46)$$
$$g_y = - g\times y/r = - g\{b/a\}\sin(E)/\{1 - e\cos(E)\} \quad (47)$$
$$g = (g_x^2 + g_y^2)^{0.5} = K_\odot/r^2 = K_\odot/[a\{1 - e\times\cos(E)\}]^2 \quad (48)$$

Here E is intermediate variable (so called "eccentric anomaly"), which is related with time as

$$E - e\times\sin(E) = 2\pi t/\tau_s \quad (49)$$

Here, as before, t = 0 corresponds initial state given by Eqs. (34-39). Eq. (49) can not be solved analytically with respect to E, and should be solved iteratively.

It should be noted that the real Sun has its own very complex trajectory with radius about 1 million of kilometers (mainly, due to influence of the Jupiter), and the analytical solutions, given above, are just approximations to reality. However, these analytical solutions are excellent tool for calibration of numerical methods.

NUMERICAL INTEGRATION

In general case, double integration of equations such as Eqs. (3, 4) is based on extrapolation of “current parameters” of state to “near future”. “Current parameters” may be marked with subscript “1”. “New parameters”, which should be guessed from current ones, may be marked with subscript “2”.

If the period of rotation τ_s (sidereal year) is known, it is convenient to express time increment $\Delta t = t_2 - t_1$ as

$$\Delta t = \tau_s/N \qquad (50)$$

Here N is total number of steps per rotation. If eccentricity of orbit ranges from 0 to 1, the orbit is ellipse, and planet should return to initial (t = 0) position at Nth step of integration.

FIRST APPROACH INTEGRATION SCHEME

One may assume, that changes of acceleration during time step $\Delta t = t_2 - t_1$ are negligible:

$$g_x \text{ (from } t_1 \text{ to } t_2) = g_{x1} \qquad (51)$$
$$g_y \text{ (from } t_1 \text{ to } t_2) = g_{y1} \qquad (52)$$

Thus new values for the components of velocity may be found via integration of Eqs.(51, 52) from t_1 to $t_2 = t_1 + \Delta t$:

$$v_{x2} = v_{x1} + g_{x1}\Delta t \qquad (53)$$
$$v_{y2} = v_{y1} + g_{y1}\Delta t \qquad (54)$$

Similarly, one may estimate new coordinates, and new distance to the Sun:

$$x_2 = x_1 + v_{x1}\Delta t + g_{x1}\Delta t^2/2 \qquad (55)$$
$$y_2 = y_1 + v_{y1}\Delta t + g_{y1}\Delta t^2/2 \qquad (56)$$
$$r_2 = (x_2^2 + y_2^2)^{0.5} \qquad (57)$$

Thus, new values for components of acceleration may be calculated from:

$$g_{x2} = -g_2\{x_2/r_2\} = -K_\odot\{x_2/r_2^3\} \qquad (58)$$
$$g_{y2} = -g_2\{y_2/r_2\} = -K_\odot\{y_2/r_2^3\} \qquad (59)$$

And now, applying $x_1 = x_2$, $y_1 = y_2$, $v_{x1} = v_{x2}$, $v_{y1} = v_{y2}$, $g_{x1} = g_{x2}$, $g_{y1} = g_{y2}$, one may return to Eqs (53-59) to perform next step of integration.

The first approach is very uncertain and not applicable in the most of cases. As may be seen in Tab. 1, error is proportional to [t, years]^2, and inversely proportional to [N, steps per year]. In the limit $e \to 1$, velocity in perihelion, v_p, and thus length of step, $v_p \times \Delta t$, approaches to infinity. Because of this, in the limit $e \to 1$, error also approaches to infinity:

$$\text{Error, km} \sim 3\times10^{10}\{1 + e/(1 - e)^{2.5}\}[\text{t, years}]^2/[\text{N, steps per year}] \qquad (60)$$

Tab. 1. First approach: maximum error in coordinates, as $\{(x - x_{exact})^2 + (y - y_{exact})^2\}^{0.5}$, km. In square brackets: duration of calculations.
Analytical model: a = 149597870.7 km ; 1 year (i.e. 1 rotation) = 31558150 s.

N, steps per year	E	Max. error for 1 year	Max. error for 10 years
100	0	219 000 000 km [~ 0 s]	516 000 000 km [~ 0 s]
1 000	0	27 700 000 km [~ 0 s]	338 000 000 km [~ 0.3 s]
10 000	0 0.1 0.2 0.3 0.4 0.5 0.6	2 840 000 km 3 200 000 3 870 000 5 030 000 [~ 0.3 s] 7 110 000 11 100 000 20 000 000	235 000 000 km [~ 1.8 s]
100 000	0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8	284 000 km 321 000 387 000 503 000 711 000 [~ 1.8 s] 1 110 000 2 010 000 4 440 000 14 100 000	27 700 000 km 31 500 000 38 200 000 49 500 000 68 700 000 [~ 17 s] 101 000 000

Tab. 2. Second approach: maximum error in coordinates, as $\{(x - x_{exact})^2 + (y - y_{exact})^2\}^{0.5}$, km.
In square brackets: duration of calculations.
Analytical model: a = 149597870.7 km ; 1 year = 31558150 s.

N, steps per year	e	Max. error for 1 year	Max. error for 10 years
100	0	621 000 km	6 210 000 km
	0.1	1 820 000 [~ 0 s]	18 200 000 [~ 0 s]
	0.2	4 090 000	40 600 000
	0.3	8 900 000	85 300 000
1 000	0	6 180 km	61 800 km
	0.1	18 100	181 000
	0.2	40 700	407 000
	0.3	88 400 [~ 0 s]	884 000 [~ 0.4 s]
	0.4	201 000	2 010 000
	0.5	507 000	5 070 000
	0.6	1 510 000	15 100 000
	0.7	5 990 000	55 500 000
10 000	0	61.8 km	618 km
	0.1	181	1 810
	0.2	407	4 070
	0.3	884	8 840
	0.4	2 010 [~ 0.4 s]	20 100 [~ 3 s]
	0.5	5 070	50 700
	0.6	15 100	151 000
	0.7	59 800	598 000
	0.8	400 000	3 990 000
	0.9	9 520 000	63 600 000
100 000	0	0.618 km	6.18 km
	0.1	1.81	18.1
	0.2	4.07	40.7
	0.3	8.84	88.4
	0.4	20.1	201
	0.5	50.7 [~ 3 s]	507 [~ 32 s]
	0.6	151	1 510
	0.7	598	5 980
	0.8	4 000	40 000
	0.9	97 000	970 000
	0.95	2 260 000	18 400 000
	0.98	50 300 000	201 000 000

Tab. 3. Third approach: maximum error in coordinates, as $\{(x - x_{exact})^2 + (y - y_{exact})^2\}^{0.5}$, km.
Numbers in square brackets: duration of calculations.
Analytical model: a = 149 597 870.7 km ; 1 year = 31558150 s.

N, steps per year	e	Max. error for 1 year	Max. error for 10 years
100	0	32.5 km	307 km
	0.1	65.1	650
	0.2	245	2 450
	0.3	678	6 770
	0.4	1 830 [~ 0 s]	18 200 [~ 0 s]
	0.5	4 820	47 500
	0.6	6 660	113 000
	0.7	220 000	2 490 000
	0.8	21 500 000	155 000 000
1 000	0	0.00326 km	0.030 7 km
	0.1	0.00651	0.064 9
	0.2	0.0245	0.245
	0.3	0.0680	0.680
	0.4	0.185	1.85
	0.5	0.504 [~ 0 s]	5.04 [~ 1 s]
	0.6	1.04	10.4
	0.7	9.60	93.9
	0.8	676	6 760
	0.9	245 000	2 470 000
	0.95	55 400 000	240 000 000
10 000	0	0.000 000 60 km	0.000 006 90 km
	0.1	0.000 002 53	0.000 639
	0.2	0.000 006 25	0.000 039
	0.3	0.000 039 1	0.000 243
	0.4	0.000 057 9	0.000 849
	0.5	0.000 020 3 [~ 1 s]	0.001 08 [~ 7.5 s]
	0.6	0.000 144	0.001 40
	0.7	0.000 916	0.010
	0.8	0.066 5	0.664
	0.9	22.2	222
	0.95	5 100	51 000
	0.98	5 720 000	29 100 000
100 000	0	0.000 003 8 km	0.000 044
	0.1	0.000 012 3	0.000 095
	0.2	0.000 051 6	0.000 486
	0.3	0.000 038 7	0.000 186
	0.4	0.000 053 8	0.002 95
	0.5	0.000 055 8	0.001 32
	0.6	0.000 063 4 [~ 7.5 s]	0.001 41 [~72 s]
	0.7	0.000 041 0	0.002 54
	0.8	0.000 163	0.000 703
	0.9	0.002 69	0.017 3
	0.95	0.506	5.07
	0.98	553	5 530
	0.99	105 000	1 040 000

SECOND APPROACH INTEGRATION SCHEME

Second approach is based on assumption that the acceleration changes linearly with time within the time step $t_2 - t_1 = \Delta t$:

$$g_x \text{ (from } t_1 \text{ to } t_2) = g_{x1} + b_{x1}\times(t - t_1) \quad (61)$$
$$g_y \text{ (from } t_1 \text{ to } t_2) = g_{y1} + b_{y1}\times(t - t_1) \quad (62)$$

Here b_{x1} and b_{y1} are some unknown coefficients. Thus, the components of acceleration at the moment of time t_2 are related with these at t_1 as:

$$g_{x2} = g_{x1} + b_{x1}\times\Delta t \quad (63)$$
$$g_{y2} = g_{y1} + b_{y1}\times\Delta t \quad (64)$$

New coordinates and new distance to the Sun may be obtained via double integration of Eqs (61, 62) from t_1 to $t_2 = t_1 + \Delta t$, with coefficients b_{x1} and b_{y1} substituted from Eqs (63, 64)

$$x_2 = x_1 + v_{x1}\Delta t + (1/2)g_{x1}\Delta t^2 + (1/6)b_{x1}\Delta t^3 = = x_1 + v_{x1}\Delta t + (1/6)\{2g_{x1} + g_{x2})\Delta t^2 \quad (65)$$
$$y_2 = y_1 + v_{y1}\Delta t + (1/2)g_{y1}\Delta t^2 + (1/6)b_{y1}\Delta t^3 = = y_1 + v_{y1}\Delta t + (1/6)\{2g_{y1} + g_{y2})\Delta t^2 \quad (66)$$
$$r_2 = (x_2{}^2+y_2{}^2)^{0.5} \quad (67)$$

At first iteration, the values g_{x2} and g_{y2} should be estimated as $g_{x2} = g_{x1}$ and $g_{y2} = g_{y1}$. With estimates for x_2, y_2 and r_2, one may obtain closer estimates for g_{x2} and g_{y2}:

$$g_2 = K_\odot/r_2{}^2 \quad (68)$$
$$g_{x2} = - g_2\{x_2/r_2\} = - K_\odot\{x_2/r_2{}^3\} \quad (69)$$
$$g_{y2} = - g_2\{y_2/r_2\} = - K_\odot\{y_2/r_2{}^3\} \quad (70)$$

Now, with new values for g_{x2} and g_{y2}, one may return to Eqs (65-70) to obtain more close estimates. Within the present scheme, one re-iteration is enough, whereas second and third re-iterations give no improvement. New values for the components of velocity may be then obtained via single integration of Eqs (61, 62) from t_1 to $t_2 = t_1 + \Delta t$:

$$v_{x2} = v_{x1} + g_{x1}\Delta t + (1/2)b_{x1}\Delta t^2 = = v_{x1} + (1/2)\{g_{x1} + g_{x2}\}\Delta t \quad (71)$$
$$v_{y2} = v_{y1} + g_{y1}\Delta t + (1/2)b_{y1}\Delta t^2 = = v_{y1} + (1/2)\{g_{y1} + g_{y2}\}\Delta t \quad (72)$$

Now, applying $x_1 = x_2$, $y_1 = y_2$, $v_{x1} = v_{x2}$, $v_{y1} = v_{y2}$, $a_{x1} = a_{x2}$, and $a_{y1} = a_{y2}$, one may return to Eqs. (65-72), and perform next step of integration. As may be seen from Tab. 2, accuracy of the second approach is ~ 3-6 orders better than for the first approach (see Tab. 1). Error is directly proportional to time (years), and inversely proportional to N^2. Like in case of first approach, in the limit $e \rightarrow 1$, error increases to infinity:

$$\text{Error, km} \sim 7\times10^9\{(1 + e)/(1 - e)\}^4[\text{t, years}]/[\text{N, steps per year}]^2 \quad (73)$$

THIRD APPROACH INTEGRATION SCHEME

Third approach is based on assumption that the acceleration changes with time (within the time step $\Delta t = t_2 - t_1$) along with parabola:

$$g_x \text{ (from } t_1 \text{ to } t_2) = g_{x1} + b_{x1}\times\{t-t_1\} + c_{x1}\times\{t-t_1\}^2 \quad (74)$$

$$g_y \text{ (from } t_1 \text{ to } t_2) = g_{y1} + b_{y1}\times\{t-t_1\} + c_{x1}\times\{t-t_1\}^2 \quad (75)$$

Here b_{x1}, b_{y1}, c_{x1}, and c_{y1} are some unknown coefficients.
Thus, the components of acceleration at time t_2 are related with these at t_1 as:

$$g_{x2} = g_{x1} + b_{x1}\times\Delta t + c_{x1}\times\Delta t^2 \quad (76)$$

$$g_{y2} = g_{y1} + b_{y1}\times\Delta t + c_{y1}\times\Delta t^2 \quad (77)$$

To obtain coefficients b and c of parabola, it is necessary to know, at least, 3 points. Therefore, in addition to times t_1 and t_2, it is necessary to consider also the "mid point", at $t_m = (t_1 + t_2)/2$. In accordance with Eqs (74, 75), components of acceleration in the midpoint are:

$$g_{xm} = g_{x1} + b_{x1}\times\{\Delta t/2\} + c_{x1}\times\{\Delta t/2\}^2 \quad (78)$$

$$g_{ym} = g_{y1} + b_{y1}\times\{\Delta t/2\} + c_{y1}\times\{\Delta t/2\}^2 \quad (79)$$

Coordinates of midpoint x_m and y_m, and distance from midpoint to the Sun r_m may be estimated via double integration of Eqs (74, 75) from $t = t_1$ to $t = t_1 + (\Delta t/2)$, with coefficients b_{x1}, b_{y1}, c_{x1} and c_{y1} substituted from Eqs (76-79):

$$x_m = x_1 + \{v_{x1}/2\}\Delta t + \{g_{x1}/8\}\Delta t^2 + \{b_{x1}/48\}\Delta t^3 + \{c_{x1}/192\}\Delta t^4 =$$
$$= x_1 + \{v_{x1}/2\}\Delta t + \{1/96\}\{7g_{x1} + 6g_{xm} - g_{x2}\}\Delta t^2 \quad (80)$$

$$y_m = y_1 + \{v_{y1}/2\}\Delta t + \{g_{y1}/8\}\Delta t^2 + \{b_{y1}/48\}\Delta t^3 + \{c_{y1}/192\}\Delta t^4 =$$
$$= y_1 + \{v_{y1}/2\}\Delta t + \{1/96\}\{7g_{y1} + 6g_{ym} - g_{y2}\}\Delta t^2 \quad (81)$$

$$r_m = (x_m^2 + y_m^2)^{0.5} \quad (82)$$

At first iteration, the values g_{xm} , g_{ym} , g_{x2} and g_{y2} in Eqs (80, 81) should be estimated as $g_{xm} = g_{x1}$, $g_{ym} = g_{y1}$, $g_{x2} = g_{x1}$ and $g_{y2} = g_{y1}$. Then, with estimates for x_m and y_m, the components of acceleration g_{xm}, g_{ym} at midpoint may be estimated from:

$$g_{xm} = -K_\odot\{x_m/r_m^3\} \quad (83)$$

$$g_{ym} = -K_\odot\{y_m/r_m^3\} \quad (84)$$

Similarly, new coordinates, x_2 and y_2, and new distance to the Sun, r_2, may be calculated from:

$$x_2 = x_1 + v_{x1}\Delta t + (1/2)g_{x1}\Delta t^2 + (1/6)b_{x1}\Delta t^3 + (1/12)c_{x1}\Delta t^4 =$$
$$= x_1 + v_{x1}\Delta t + (1/6)\{g_{x1} + 2g_{xm}\}\Delta t^2 \quad (85)$$

$$y_2 = y_1 + v_{y1}\Delta t + (1/2)g_{y1}\Delta t^2 + (1/6)b_{y1}\Delta t^3 + (1/12)c_{y1}\Delta t^4 =$$
$$= y_1 + v_{y1}\Delta t + (1/6)\{g_{y1} + 2g_{ym})\Delta t^2 \quad (86)$$

$$r_2 = (x_2^2+y_2^2)^{0.5} \quad (87)$$

With these values, new components of acceleration may be found from:

$$g_{x2} = - K_{\odot}\{x_2/r_2^3\} \quad (88)$$
$$g_{y2} = - K_{\odot}\{y_2/r_2^3\} \quad (89)$$

Now, with estimates for g_{xm} , g_{ym} , g_{x2} and g_{y2}, it is necessary to perform re-iteration, beginning from Eq. (80), in order to obtain more close values. Within the present scheme, two re-iterations are enough, whereas third and forth re-iterations give no improvement.

New components of velocity may be then calculated as:

$$v_{x2} = v_{x1} + g_{x1}\Delta t + (1/2)b_{x1}\Delta t^2 + (1/3)\,c_{x1}\Delta t^3 = v_{x1} + (1/6)\{g_{x1} + 4g_{xm} + g_{x2}\}\Delta t \quad (90)$$
$$v_{y2} = v_{y1} + g_{y1}\Delta t + (1/2)b_{y1}\Delta t^2 + (1/3)\,c_{y1}\Delta t^3 = v_{y1} + (1/6)\{g_{y1} + 4g_{ym} + g_{y2}\}\Delta t \quad (91)$$

Now, applying $x_1 = x_2$, $y_1 = y_2$, $v_{x1} = v_{x2}$, $v_{y1} = v_{y2}$, $g_{x1} = g_{x2}$, and $g_{y1} = g_{y2}$, $g_{xm} = g_{x2}$, and $g_{ym} = g_{y2}$, one may return to Eqs. (80-91), and perform next step of integration.

As may be seen from Tab. 3, accuracy of the third approach is ~ 3-6 orders better than for the second approach (see Tab. 2). Error is directly proportional to time (years), inversely proportional to N^4. Like in case of first and second approaches, in the limit $e \to 1$, error increases to infinity. Due to significant contribution from rounding errors, uncertainty oscillates with eccentricity and increases with N at N > 1/10000 at low eccentricities. In general, error is consistent with relation:

$$\text{Error, km} \sim 3\times10^9\times[\text{t, years}]/\{(1-e)^8[\text{N, steps per year}]^4\} + 3\times10^{-11}\times[\text{t, years}]\times[\text{N, steps per year}]\times\{1 + 50\times e\} \quad (92)$$

Note that the second term in Eq. (92) reflects the rounding error for calculations with double accuracy (i.e. with 16 digits).

CONCLUDING REMARKS

Even with time step 1/1000 of rotation ("year"), third approach is efficient method for calculation of planetary orbits. However, for the orbits of comets, third approach is applicable solely with time step 1/1000000 of rotation and smaller. In general, third approach integration scheme is almost exact method for majority of applications.

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